UNIVERSIDADE FEDERAL DE PERNAMBUCO
CENTRO DE CIÊNCIAS SOCIAIS APLICADAS
DEPARTAMENTO DE ECONOMIA

PIMES – PROGRAMA DE PÓS-GRADUAÇÃO EM ECONOMIA
Disciplina: MATEMÁTICA I

1ª LISTA DE EXERCÍCIOS

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IV. Which of the following is true about the system given below?

$$\begin{aligned} x + y + z &= 3 \\ 2x + 2y + 2z &= 6 \\ 3x + 3y + 3z &= 10 \end{aligned}$$

a. It has the unique solution $x = 1$, $y = 1$, $z = 1$.
b. It is inconsistent.
c. It has an infinite number of solutions.

In Problems 1–20 use Gauss-Jordan or Gaussian elimination to find all solutions, if any, to the given systems.

1. $$\begin{aligned} x_1 - 2x_2 + 3x_3 &= 11 \\ 4x_1 + x_2 - x_3 &= 4 \\ 2x_1 - x_2 + 3x_3 &= 10 \end{aligned}$$

2. $$\begin{aligned} -2x_1 + x_2 + 6x_3 &= 18 \\ 5x_1 + 8x_3 &= -16 \\ 3x_1 + 2x_2 - 10x_3 &= -3 \end{aligned}$$

3. $$\begin{aligned} 3x_1 + 6x_2 - 6x_3 &= 9 \\ 2x_1 - 5x_2 + 4x_3 &= 6 \\ -x_1 + 16x_2 - 14x_3 &= -3 \end{aligned}$$

4. $$\begin{aligned} 3x_1 + 6x_2 - 6x_3 &= 9 \\ 2x_1 - 5x_2 + 4x_3 &= 6 \\ 5x_1 + 28x_2 - 26x_3 &= -8 \end{aligned}$$

5. $$\begin{aligned} x_1 + x_2 - x_3 &= 7 \\ 4x_1 - x_2 + 5x_3 &= 4 \\ 2x_1 + 2x_2 - 3x_3 &= 0 \end{aligned}$$

6. $$\begin{aligned} x_1 + x_2 - x_3 &= 7 \\ 4x_1 - x_2 + 5x_3 &= 4 \\ 6x_1 + x_2 + 3x_3 &= 18 \end{aligned}$$

7. $$\begin{aligned} x_1 + x_2 - x_3 &= 7 \\ 4x_1 - x_2 + 5x_3 &= 4 \\ 6x_1 + x_2 + 3x_3 &= 20 \end{aligned}$$

8. $$\begin{aligned} x_1 - 2x_2 + 3x_3 &= 0 \\ 4x_1 + x_2 - x_3 &= 0 \\ 2x_1 - x_2 + 3x_3 &= 0 \end{aligned}$$

9. $$\begin{aligned} x_1 + x_2 - x_3 &= 0 \\ 4x_1 - x_2 + 5x_3 &= 0 \\ 6x_1 + x_2 + 3x_3 &= 0 \end{aligned}$$

10. $$\begin{aligned} 2x_2 + 5x_3 &= 6 \\ x_1 - 2x_3 &= 4 \\ 2x_1 + 4x_2 &= -2 \end{aligned}$$

11. $$\begin{aligned} x_1 + 2x_2 - x_3 &= 4 \\ 3x_1 + 4x_2 - 2x_3 &= 7 \end{aligned}$$

12. $$\begin{aligned} x_1 + 2x_2 - 4x_3 &= 4 \\ -2x_1 - 4x_2 + 8x_3 &= -8 \end{aligned}$$

13. $$\begin{aligned} x_1 + 2x_2 - 4x_3 &= 4 \\ -2x_1 - 4x_2 + 8x_3 &= -9 \end{aligned}$$

14. $$\begin{aligned} x_1 + 2x_2 - x_3 + x_4 &= 7 \\ 3x_1 + 6x_2 - 3x_3 + 3x_4 &= 21 \end{aligned}$$

15. $$\begin{aligned} 2x_1 + 6x_2 - 4x_3 + 2x_4 &= 4 \\ x_1 - x_3 + x_4 &= 5 \\ -3x_1 + 2x_2 - 2x_3 &= -2 \end{aligned}$$

16. $$\begin{aligned} x_1 - 2x_2 + x_3 + x_4 &= 2 \\ 3x_1 + 2x_3 - 2x_4 &= -8 \\ 4x_2 - x_3 - x_4 &= 1 \\ -x_1 + 6x_2 - 2x_3 &= 7 \end{aligned}$$

17. $$\begin{aligned} x_1 - 2x_2 + x_3 + x_4 &= 2 \\ 3x_1 + 2x_3 - 2x_4 &= -8 \\ 4x_2 - x_3 - x_4 &= 1 \\ 5x_1 + 3x_3 - x_4 &= -3 \end{aligned}$$

18. $$\begin{aligned} x_1 - 2x_2 + x_3 + x_4 &= 2 \\ 3x_1 + 2x_3 - 2x_4 &= -8 \\ 4x_2 - x_3 - x_4 &= 1 \\ 5x_1 + 3x_3 - x_4 &= 0 \end{aligned}$$

19. $$\begin{aligned} x_1 + x_2 &= 4 \\ 2x_1 - 3x_2 &= 7 \\ 3x_1 + 2x_2 &= 8 \end{aligned}$$

20. $$\begin{aligned} x_1 + x_2 &= 4 \\ 2x_1 - 3x_2 &= 7 \\ 3x_1 - 2x_2 &= 11 \end{aligned}$$

36. In the Leontief input-output model of Example 8 suppose that there are three industries. Suppose further that $e_1 = 10$, $e_2 = 15$, $e_3 = 30$, $a_{11} = \frac{1}{3}$, $a_{12} = \frac{1}{2}$, $a_{13} = \frac{1}{6}$, $a_{21} = \frac{1}{4}$, $a_{22} = \frac{1}{4}$, $a_{23} = \frac{1}{8}$, $a_{31} = \frac{1}{12}$, $a_{32} = \frac{1}{3}$, and $a_{33} = \frac{1}{6}$. Find the output of each industry such that supply exactly equals demand.

37. In Example 10 assume that there are 15,000 units of the first food, 10,000 units of the second, and 35,000 units of the third supplied to the lake each week. Assuming that all three foods are consumed, what populations of the three species can coexist in the lake? Is there a unique solution?

38. A traveler who just returned from Europe spent \$30 a day for housing in England, \$20 a day in France, and \$20 a day in Spain. For food the traveler spent \$20 a day in England, \$30 a day in France, and \$20 a day in Spain. The traveler spent \$10 a day in each country for incidental expenses. The traveler's records of the trip indicate a total of \$340 spent for housing, \$320 for food, and \$140 for incidental expenses while traveling in these countries. Calculate the number of days the traveler spent in each of the countries or show that the records must be incorrect because the amounts spent are incompatible with each other.

39. An investor remarks to a stockbroker that all her stock holdings are in three companies, Eastern Airlines, Hilton Hotels, and McDonald's, and that 2 days ago the value of her stocks went down \$350 but yesterday the value increased by \$600. The broker recalls that 2 days ago the price of Eastern Airlines stock dropped by \$1 a share, Hilton Hotels dropped \$1.50, but the price of McDonald's stock rose by \$0.50. The broker also remembers that yesterday the price of Eastern Airlines stock rose \$1.50, there was a further drop of \$0.50 a share in Hilton Hotels stock, and McDonald's stock rose \$1. Show that the broker does not have enough information to calculate the number of shares the investor owns of each company's stock, but that when the investor says that she owns 200 shares of McDonald's stock, the broker can calculate the number of shares of Eastern Airlines and Hilton Hotels.

40. An intelligence agent knows that 60 aircraft, consisting of fighter planes and bombers, are stationed at a certain secret airfield. The agent wishes to determine how many of the 60 are fighter planes and how many are bombers. There is a type of rocket carried by both sorts of planes; the fighter carries six of these rockets, the bomber only two. The agent learns that 250 rockets are required to arm every plane at this airfield. Furthermore, the agent overhears a remark that there are twice as many fighter planes as bombers at the base (that is, the number of fighter planes minus twice the number of bombers equals zero). Calculate the number of fighter planes and bombers at the airfield or show that the agent's information must be incorrect, because it is inconsistent.

41. Consider the system

$$\begin{aligned} 2x_1 - x_2 + 3x_3 &= a \\ 3x_1 + x_2 - 5x_3 &= b \\ -5x_1 - 5x_2 + 21x_3 &= c \end{aligned}$$

Show that the system is inconsistent if $c \neq 2a - 3b$.

42. Consider the system

$$\begin{aligned} 2x_1 + 3x_2 - x_3 &= a \\ x_1 - x_2 + 3x_3 &= b \\ 3x_1 + 7x_2 - 5x_3 &= c \end{aligned}$$

Find conditions on a, b, and c such that the system is consistent.

*43. Consider the general system of three linear equations in three unknowns:

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= b_1 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= b_2 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= b_3 \end{aligned}$$

Find conditions on the coefficients a_{ij} such that the system has a unique solution.

PROBLEMS 6.4

SELF-QUIZ

I. Which of the following systems *must* have nontrivial solutions?

a. $a_{11}x_1 + a_{12}x_2 = 0$
$a_{21}x_1 + a_{22}x_2 = 0$

b. $a_{11}x_1 + a_{12}x_2 = 0$
$a_{21}x_1 + a_{22}x_2 = 0$
$a_{31}x_1 + a_{32}x_2 = 0$

c. $a_{11}x_1 + a_{12}x_2 + a_{13}x_3 = 0$
$a_{21}x_1 + a_{22}x_2 + a_{23}x_3 = 0$

II. For what value of k will the following system have nontrivial solutions?

$$\begin{aligned} x + y + z &= 0 \\ 2x + 3y + 4z &= 0 \\ 3x + 4y + kz &= 0 \end{aligned}$$

a. 1 **b.** 2 **c.** 3
d. 4 **e.** 5 **f.** 0

In Problems 1–13 find all solutions to the homogeneous systems.

1. $2x_1 - x_2 = 0$
$3x_1 + 4x_2 = 0$

2. $x_1 - 5x_2 = 0$
$-x_1 + 5x_2 = 0$

3. $x_1 + x_2 - x_3 = 0$
$2x_1 - 4x_2 + 3x_3 = 0$
$3x_1 + 7x_2 - x_3 = 0$

4. $x_1 + x_2 - x_3 = 0$
$2x_1 - 4x_2 + 3x_3 = 0$
$-x_1 - 7x_2 + 6x_3 = 0$

5. $x_1 + x_2 - x_3 = 0$
$2x_1 - 4x_2 + 3x_3 = 0$
$-5x_1 + 13x_2 - 10x_3 = 0$

6. $2x_1 + 3x_2 - x_3 = 0$
$6x_1 - 5x_2 + 7x_3 = 0$

7. $4x_1 - x_2 = 0$
$7x_1 + 3x_2 = 0$
$-8x_1 + 6x_2 = 0$

8. $x_1 - x_2 + 7x_3 - x_4 = 0$
$2x_1 + 3x_2 - 8x_3 + x_4 = 0$

9. $x_1 - 2x_2 + x_3 + x_4 = 0$
$3x_1 + 2x_3 - 2x_4 = 0$
$4x_2 - x_3 - x_4 = 0$
$5x_1 + 3x_3 - x_4 = 0$

10. $-2x_1 + 7x_4 = 0$
$x_1 + 2x_2 - x_3 + 4x_4 = 0$
$3x_1 - x_3 + 5x_4 = 0$
$4x_1 + 2x_2 + 3x_3 = 0$

11. $2x_1 - x_2 = 0$
$3x_1 + 5x_2 = 0$
$7x_1 - 3x_2 = 0$
$-2x_1 + 3x_2 = 0$

12. $x_1 - 3x_2 = 0$
$-2x_1 + 6x_2 = 0$
$4x_1 - 12x_2 = 0$

13. $x_1 + x_2 - x_3 = 0$
$4x_1 - x_2 + 5x_3 = 0$
$-2x_1 + x_2 - 2x_3 = 0$
$3x_1 + 2x_2 - 6x_3 = 0$

14. Show that the homogeneous system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 &= 0 \\ a_{21}x_1 + a_{22}x_2 &= 0 \end{aligned}$$

has an infinite number of solutions if and only if $a_{11}a_{22} - a_{12}a_{21} = 0$.

15. Consider the system

$$\begin{aligned} 2x_1 - 3x_2 + 5x_3 &= 0 \\ -x_1 + 7x_2 - x_3 &= 0 \\ 4x_1 - 11x_2 + kx_3 &= 0 \end{aligned}$$

For what value of k will the system have nontrivial solutions?

***16.** Consider the 3×3 homogeneous system

$$\begin{aligned} a_{11}x_1 + a_{12}x_2 + a_{13}x_3 &= 0 \\ a_{21}x_1 + a_{22}x_2 + a_{23}x_3 &= 0 \\ a_{31}x_1 + a_{32}x_2 + a_{33}x_3 &= 0 \end{aligned}$$

Find conditions on the coefficients a_{ij} such that the zero solution is the only solution.

ANSWERS TO SELF-QUIZ

I. c **II.** e

PROBLEMS 6.5

I. Which of the following is true of the matrix

$$\begin{pmatrix} 1 & 2 & 3 \\ 7 & -1 & 0 \end{pmatrix}?$$

a. It is a square matrix.
b. If multiplied by the scalar -1, the product is

$$\begin{pmatrix} -1 & -2 & -3 \\ -7 & 1 & 0 \end{pmatrix}.$$

c. It is a 3×2 matrix.
d. It is the sum of $\begin{pmatrix} 3 & 1 & 4 \\ 7 & 2 & 0 \end{pmatrix}$ and

$$\begin{pmatrix} -2 & 1 & 1 \\ 0 & 1 & 0 \end{pmatrix}.$$

II. Which of the following is $2A - 4B$ if $A = (2 \quad 0 \quad 0)$ and $B = (3 \quad 1)$?
a. $(-8 \quad -4)$
b. $(5 \quad 0 \quad 1)$
c. $(16 \quad -4 \quad 0)$
d. This operation cannot be performed.

III. Which of the following is true when finding the difference of two matrices?
a. The matrices must have the same size.
b. The matrices must be square.
c. The matrices must both be row vectors or both be column vectors.
d. One matrix must be a row vector and the other must be a column vector.

IV. Which of the following would be the entries in the second column of matrix B, if

$$\begin{pmatrix} 3 & -4 & 0 \\ 2 & 8 & -1 \end{pmatrix} + B = \begin{pmatrix} 0 & 0 & 0 \\ 0 & 0 & 0 \end{pmatrix}?$$

a. $-2, -8, 1$ **b.** $4, -8$
c. $2, 8, -1$ **d.** $-4, 8$

V. Which of the following must be the second row of matrix B if $3A - B = 2C$ for

$$A = \begin{pmatrix} 1 & -1 & 1 \\ 0 & 0 & 3 \\ 4 & 2 & 0 \end{pmatrix} \text{ and } C = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}?$$

a. $-3 \quad 2 \quad 6$ **b.** $0 \quad -2 \quad 9$
c. $3 \quad -2 \quad 6$ **d.** $0 \quad 2 \quad -9$

In Problems 1–12 perform the indicated computation with $A = \begin{pmatrix} 1 & 3 \\ 2 & 5 \\ -1 & 2 \end{pmatrix}$, $B = \begin{pmatrix} -2 & 0 \\ 1 & 4 \\ -7 & 5 \end{pmatrix}$, and $C = \begin{pmatrix} -1 & 1 \\ 4 & 6 \\ -7 & 3 \end{pmatrix}$.

1. $3A$ **2.** $A + B$
3. $A - C$ **4.** $2C - 5A$
5. $0B$ (0 is the scalar zero) **6.** $-7A + 3B$
7. $A + B + C$ **8.** $C - A - B$
9. $2A - 3B + 4C$ **10.** $7C - B + 2A$
11. Find a matrix D such that $2A + B - D$ is the 3×2 zero matrix.
12. Find a matrix E such that $A + 2B - 3C + E$ is the 3×2 zero matrix.

In Problems 13–20 perform the indicated computation with $A = \begin{pmatrix} 1 & -1 & 2 \\ 3 & 4 & 5 \\ 0 & 1 & -1 \end{pmatrix}$, $B = \begin{pmatrix} 0 & 2 & 1 \\ 3 & 0 & 5 \\ 7 & -6 & 0 \end{pmatrix}$, and $C = \begin{pmatrix} 0 & 0 & 2 \\ 3 & 1 & 0 \\ 0 & -2 & 4 \end{pmatrix}$.

13. $A - 2B$ **14.** $3A - C$
15. $A + B + C$ **16.** $2A - B + 2C$
17. $C - A - B$ **18.** $4C - 2B + 3A$
19. Find a matrix D such that $A + B + C + D$ is the 3×3 zero matrix.
20. Find a matrix E such that $3C - 2B + 8A - 4E$ is the 3×3 zero matrix.
21. Let $A = (a_{ij})$ be an $m \times n$ matrix and let $\bar{0}$ denote the $m \times n$ zero matrix. Use appropriate definitions to show that $0A = \bar{0}$ and $\bar{0} + A = A$. Similarly, show that $1A = A$.
22. If $A = (a_{ij})$, $B = (b_{ij})$, and $C = (c_{ij})$ are $m \times n$

PROBLEMS 6.6

SELF-QUIZ

I. Which of the following is true of matrix multiplication of matrices A and B?

a. It can be performed only if A and B are square matrices.
b. Each entry c_{ij} is the product of a_{ij} and b_{ij}.
c. $AB = BA$.
d. It can be performed only if the number of columns of A is equal to the number of rows of B.

II. Which of the following would be the size of the product matrix AB when a 2×4 matrix A is multiplied by a 4×3 matrix B?

a. 2×3
b. 3×2
c. 4×4
d. This product cannot be found.

III. Which of the following is true of matrices A and B if AB is a column vector?

a. B is a column vector.
b. A is a row vector.
c. A and B are square vectors.
d. The number of rows in A must equal the number of columns in B.

IV. Which of the following is true about a product AB if A is a 4×5 matrix?

a. B must have 4 rows and the result will have 5 columns.
b. B must have 5 columns and the result will be a square matrix.
c. B must have 4 columns and the result will have 5 rows.
d. B must have 5 rows and the result will have 4 rows.

In Problems 1–15 perform the indicated computation.

1. $\begin{pmatrix} 2 & 3 \\ -1 & 2 \end{pmatrix}\begin{pmatrix} 4 & 1 \\ 0 & 6 \end{pmatrix}$

2. $\begin{pmatrix} 3 & -2 \\ 1 & 4 \end{pmatrix}\begin{pmatrix} -5 & 6 \\ 1 & 3 \end{pmatrix}$

3. $\begin{pmatrix} 1 & -1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} -1 & 0 \\ 2 & 3 \end{pmatrix}$

4. $\begin{pmatrix} -5 & 6 \\ 1 & 3 \end{pmatrix}\begin{pmatrix} 3 & -2 \\ 1 & 4 \end{pmatrix}$

5. $\begin{pmatrix} -4 & 5 & 1 \\ 0 & 4 & 2 \end{pmatrix}\begin{pmatrix} 3 & -1 & 1 \\ 5 & 6 & 4 \\ 0 & 1 & 2 \end{pmatrix}$

6. $\begin{pmatrix} 7 & 1 & 4 \\ 2 & -3 & 5 \end{pmatrix}\begin{pmatrix} 1 & 6 \\ 0 & 4 \\ -2 & 3 \end{pmatrix}$

7. $\begin{pmatrix} 1 & 6 \\ 0 & 4 \\ -2 & 3 \end{pmatrix}\begin{pmatrix} 7 & 1 & 4 \\ 2 & -3 & 5 \end{pmatrix}$

8. $\begin{pmatrix} 1 & 4 & -2 \\ 3 & 0 & 4 \end{pmatrix}\begin{pmatrix} 0 & 1 \\ 2 & 3 \end{pmatrix}$

9. $\begin{pmatrix} 1 & 4 & 6 \\ -2 & 3 & 5 \\ 1 & 0 & 4 \end{pmatrix}\begin{pmatrix} 2 & -3 & 5 \\ 1 & 0 & 6 \\ 2 & 3 & 1 \end{pmatrix}$

10. $\begin{pmatrix} 2 & -3 & 5 \\ 1 & 0 & 6 \\ 2 & 3 & 1 \end{pmatrix}\begin{pmatrix} 1 & 4 & 6 \\ -2 & 3 & 5 \\ 1 & 0 & 4 \end{pmatrix}$

11. $(1 \quad 4 \quad 0 \quad 2)\begin{pmatrix} 3 & -6 \\ 2 & 4 \\ 1 & 0 \\ -2 & 3 \end{pmatrix}$

12. $\begin{pmatrix} 1 \\ 3 \\ 5 \end{pmatrix}(2 \quad -1 \quad 4)$

13. $\begin{pmatrix} 3 \\ -1 \\ 10 \\ 2 \end{pmatrix}(1 \quad 5 \quad -3 \quad 8)$

14. $\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}\begin{pmatrix} 3 & -2 & 1 \\ 4 & 0 & 6 \\ 5 & 1 & 9 \end{pmatrix}$

15. $\begin{pmatrix} a & b & c \\ d & e & f \\ g & h & j \end{pmatrix}\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$, where $a, b, c, d, e, f, g, h, j$ are real numbers.

16. Find a matrix $A = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ such that $A\begin{pmatrix} 2 & 3 \\ 1 & 2 \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

17. Let a_{11}, a_{12}, a_{21}, and a_{22} be given real numbers such that $a_{11}a_{22} - a_{12}a_{21} \neq 0$. Find numbers b_{11}, b_{12}, b_{21}, and b_{22} such that $\begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}\begin{pmatrix} b_{11} & b_{12} \\ b_{21} & b_{22} \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$.

18. Verify the associative law for multiplication for the matrices $A = \begin{pmatrix} 2 & -1 & 4 \\ 1 & 0 & 6 \end{pmatrix}$,

$$B = \begin{pmatrix} 1 & 0 & 1 \\ 2 & -1 & 2 \\ 3 & -2 & 0 \end{pmatrix}, \text{ and } C = \begin{pmatrix} 1 & 6 \\ -2 & 4 \\ 0 & 5 \end{pmatrix}.$$

19. As in Example 3, suppose that a group of people have contracted a contagious disease. These persons have contacts with a second group who in turn have contacts with a third group. Let $A = \begin{pmatrix} 1 & 0 & 1 & 0 \\ 0 & 1 & 1 & 0 \\ 1 & 0 & 0 & 1 \end{pmatrix}$ represent the contacts between the contagious group and the members of group 2, and let

$$B = \begin{pmatrix} 1 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 1 \end{pmatrix}$$

represent the contacts between groups 2 and 3.
(a) How many people are in each group?
(b) Find the matrix of indirect contacts between groups 1 and 3.

20. Answer the questions of Problem 19 for $A = \begin{pmatrix} 1 & 0 & 1 & 1 & 0 \\ 0 & 1 & 0 & 1 & 1 \end{pmatrix}$ and

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 & 0 & 1 \\ 0 & 1 & 0 & 1 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 1 & 1 & 1 \\ 0 & 0 & 0 & 1 & 1 & 0 & 1 \\ 0 & 1 & 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

21. A company pays its executives a salary and gives them shares of its stock as an annual bonus. Last year, the president of the company received \$80,000 and 50 shares of stock, each of the three vice-presidents was paid \$45,000 and 20 shares of stock, and the treasurer was paid \$40,000 and 10 shares of stock.
(a) Express the payments to the executives in money and stock by means of a 2×3 matrix.
(b) Express the number of executives of each rank by means of a column vector.
(c) Use matrix multiplication to calculate the total amount of money and the total number of shares of stock the company paid these executives last year.

22. Sales, unit gross profits, and unit taxes for sales of a large corporation are given in the table below. Find a matrix that shows total profits and taxes in each of the four months.

23. Let A be a square matrix. Then A^2 is defined simply as AA. Calculate $\begin{pmatrix} 2 & -1 \\ 4 & 6 \end{pmatrix}^2$.

24. Calculate A^2, where $A = \begin{pmatrix} 1 & -2 & 4 \\ 2 & 0 & 3 \\ 1 & 1 & 5 \end{pmatrix}$.

25. Calculate A^3, where $A = \begin{pmatrix} -1 & 2 \\ 3 & 4 \end{pmatrix}$.

26. Calculate A^2, A^3, A^4, and A^5, where

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 \end{pmatrix}.$$

TABLE FOR PROBLEM 22

	Product					
		Sales of Item				
Month	i	ii	iii	Item	Unit Profit (in hundreds of dollars)	Unit Taxes (in hundreds of dollars)
January	4	2	20	i	3.5	1.5
February	6	1	9	ii	2.75	2
March	5	3	12	iii	1.5	0.6
April	8	2.5	20			

27. Calculate A^2, A^3, A^4, and A^5, where

$$A = \begin{pmatrix} 0 & 1 & 0 & 0 & 0 \\ 0 & 0 & 1 & 0 & 0 \\ 0 & 0 & 0 & 1 & 0 \\ 0 & 0 & 0 & 0 & 1 \\ 0 & 0 & 0 & 0 & 0 \end{pmatrix}.$$

28. An $n \times n$ matrix A has the property that AB is the zero matrix for any $n \times n$ matrix B. Prove that A is the zero matrix.

29. A **probability matrix** is a square matrix having two properties: (i) every component is nonnegative (≥ 0) and (ii) the sum of the elements in each row is 1. The following are probability matrices:

$$P = \begin{pmatrix} \frac{1}{3} & \frac{1}{3} & \frac{1}{3} \\ \frac{1}{4} & \frac{1}{2} & \frac{1}{4} \\ 0 & 0 & 1 \end{pmatrix} \quad \text{and} \quad Q = \begin{pmatrix} \frac{1}{6} & \frac{1}{6} & \frac{2}{3} \\ 0 & 1 & 0 \\ \frac{1}{5} & \frac{1}{5} & \frac{3}{5} \end{pmatrix}$$

Show that PQ is a probability matrix.

***30.** Let P be a probability matrix. Show that P^2 is a probability matrix.

****31.** Let P and Q be probability matrices of the same size. Prove that PQ is a probability matrix.

32. Prove formula (4) by using the associative law [equation (3)].

***33.** A round robin tennis tournament can be organized in the following way. Each of the n players plays all the others, and the results are recorded in an $n \times n$ matrix R as follows:

$$R_{ij} = \begin{cases} 1 & \text{if the } i\text{th player beats the } j\text{th player} \\ 0 & \text{if the } i\text{th player loses to the } j\text{th player} \\ 0 & \text{if } i = j \end{cases}$$

The ith player is then assigned the score

$$S_i = \sum_{j=1}^{n} R_{ij} + \frac{1}{2} \sum_{j=1}^{n} (R^2)_{ij}^{\dagger}$$

(a) In a tournament between four players

$$R = \begin{pmatrix} 0 & 1 & 0 & 0 \\ 0 & 0 & 1 & 1 \\ 1 & 0 & 0 & 0 \\ 1 & 0 & 1 & 0 \end{pmatrix}.$$

Rank the players according to their scores.

(b) Interpret the meaning of the score.

34. Let O be the $m \times n$ zero matrix and let A be an $n \times p$ matrix. Show that $OA = O_1$, where O_1 is the $m \times p$ zero matrix.

35. Verify the distributive law [equation (5)] for the matrices

$$A = \begin{pmatrix} 1 & 2 & 4 \\ 3 & -1 & 0 \end{pmatrix} \quad B = \begin{pmatrix} 2 & 7 \\ -1 & 4 \\ 6 & 0 \end{pmatrix}$$

$$C = \begin{pmatrix} -1 & 2 \\ 3 & 7 \\ 4 & 1 \end{pmatrix}.$$

ANSWERS TO SELF-QUIZ

I. d **II.** a **III.** a **IV.** d

In Problems 1–6 write the given system in the form $A\mathbf{x} = \mathbf{b}$.

1. $\begin{aligned} 2x_1 - x_2 &= 3 \\ 4x_1 + 5x_2 &= 7 \end{aligned}$

2. $\begin{aligned} x_1 - x_2 + 3x_3 &= 11 \\ 4x_1 + x_2 - x_3 &= -4 \\ 2x_1 - x_2 + 3x_3 &= 10 \end{aligned}$

3. $\begin{aligned} 3x_1 + 6x_2 - 7x_3 &= 0 \\ 2x_1 - x_2 + 3x_3 &= 1 \end{aligned}$

4. $\begin{aligned} 4x_1 - x_2 + x_3 - x_4 &= -7 \\ 3x_1 + x_2 - 5x_3 + 6x_4 &= 8 \\ 2x_1 - x_2 + x_3 &= 9 \end{aligned}$

5. $\begin{aligned} x_2 - x_3 &= 7 \\ x_1 + x_3 &= 2 \\ 3x_1 + 2x_2 &= -5 \end{aligned}$

6. $\begin{aligned} 2x_1 + 3x_2 - x_3 &= 0 \\ -4x_1 + 2x_2 + x_3 &= 0 \\ 7x_1 + 3x_2 - 9x_3 &= 0 \end{aligned}$

In Problems 7–15 write out the system of equations represented by the given augmented matrix

7. $\left(\begin{array}{ccc|c} 1 & 1 & -1 & 7 \\ 4 & -1 & 5 & 4 \\ 6 & 1 & 3 & 20 \end{array}\right)$.

8. $\left(\begin{array}{cc|c} 0 & 1 & 2 \\ 1 & 0 & 3 \end{array}\right)$

9. $\left(\begin{array}{ccc|c} 2 & 0 & 1 & 2 \\ -3 & 4 & 0 & 3 \\ 0 & 5 & 6 & 5 \end{array}\right)$

10. $\left(\begin{array}{ccc|c} 2 & 3 & 1 & 2 \\ 0 & 4 & 1 & 3 \\ 0 & 0 & 0 & 0 \end{array}\right)$

11. $\left(\begin{array}{cccc|c} 1 & 0 & 0 & 0 & 2 \\ 0 & 1 & 0 & 0 & 3 \\ 0 & 0 & 1 & 0 & -5 \\ 0 & 0 & 0 & 1 & 6 \end{array}\right)$

12. $\left(\begin{array}{ccc|c} 2 & 3 & 1 & 0 \\ 4 & -1 & 5 & 0 \\ 3 & 6 & -7 & 0 \end{array}\right)$

13. $\left(\begin{array}{ccc|c} 6 & 2 & 1 & 2 \\ -2 & 3 & 1 & 4 \\ 0 & 0 & 0 & 2 \end{array}\right)$

14. $\left(\begin{array}{ccc|c} 3 & 1 & 5 & 6 \\ 2 & 3 & 2 & 4 \end{array}\right)$

15. $\left(\begin{array}{cc|c} 7 & 2 & 1 \\ 3 & 1 & 2 \\ 6 & 9 & 3 \end{array}\right)$

16. Find a matrix A and vectors $\mathbf{x}$ and $\mathbf{b}$ such that the system represented by the following augmented matrix can be written in the form $A\mathbf{x} = \mathbf{b}$ and solve the system.

$$\left(\begin{array}{ccc|c} 2 & 0 & 0 & 3 \\ 0 & 4 & 0 & 5 \\ 0 & 0 & -5 & 2 \end{array}\right)$$

In Problems 17–22 find all solutions to the given nonhomogeneous system by first finding one solution (if possible) and then finding all solutions to the associated homogeneous system.

17. $\begin{aligned} x_1 - 3x_2 &= 2 \\ -2x_1 + 6x_2 &= -4 \end{aligned}$

18. $\begin{aligned} x_1 - x_2 + x_3 &= 6 \\ 3x_1 - 3x_2 + 3x_3 &= 18 \end{aligned}$

19. $\begin{aligned} x_1 - x_2 - x_3 &= 2 \\ 2x_1 + x_2 + 2x_3 &= 4 \\ x_1 - 4x_2 - 5x_3 &= 2 \end{aligned}$

20. $\begin{aligned} x_1 - x_2 - x_3 &= 2 \\ 2x_1 + x_2 + 2x_3 &= 4 \\ x_1 - 4x_2 - 5x_3 &= 3 \end{aligned}$

21. $\begin{aligned} x_1 + x_2 - x_3 + 2x_4 &= 3 \\ 3x_1 + 2x_2 + x_3 - x_4 &= 5 \end{aligned}$

22. $\begin{aligned} x_1 - x_2 + x_3 - x_4 &= -2 \\ -2x_1 + 3x_2 - x_3 + 2x_4 &= 5 \\ 4x_1 - 2x_2 + 2x_3 - 3x_4 &= 6 \end{aligned}$

23. Consider the linear, homogeneous second-order differential equation

$$y''(x) + a(x)y'(x) + b(x)y(x) = 0 \qquad (7)$$

where $a(x)$ and $b(x)$ are continuous and the unknown function y is assumed to have a second derivative. Show that if y_1 and y_2 are solutions to (7), then $c_1y_1 + c_2y_2$ is a solution for any constants c_1 and c_2.

24. Suppose that y_p and y_q are solutions to the nonhomogeneous equation

$$y''(x) + a(x)y'(x) + b(x)y(x) = f(x) \qquad (8)$$

Show that $y_p - y_q$ is a solution to (7).

ANSWER TO SELF-QUIZ

I. d

In Problems 1–15 determine whether the given matrix is invertible. If it is, calculate the inverse.

1. $\begin{pmatrix} 2 & 1 \\ 3 & 2 \end{pmatrix}$

2. $\begin{pmatrix} -1 & 6 \\ 2 & -12 \end{pmatrix}$

3. $\begin{pmatrix} 0 & 1 \\ 1 & 0 \end{pmatrix}$

4. $\begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix}$

5. $\begin{pmatrix} a & a \\ b & b \end{pmatrix}$

6. $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 2 & 3 \\ 5 & 5 & 1 \end{pmatrix}$

7. $\begin{pmatrix} 3 & 2 & 1 \\ 0 & 2 & 2 \\ 0 & 0 & -1 \end{pmatrix}$

8. $\begin{pmatrix} 1 & 1 & 1 \\ 0 & 1 & 1 \\ 0 & 0 & 1 \end{pmatrix}$

9. $\begin{pmatrix} 1 & 6 & 2 \\ -2 & 3 & 5 \\ 7 & 12 & -4 \end{pmatrix}$

10. $\begin{pmatrix} 3 & 1 & 0 \\ 1 & -1 & 2 \\ 1 & 1 & 1 \end{pmatrix}$

11. $\begin{pmatrix} 2 & -1 & 4 \\ -1 & 0 & 5 \\ 19 & -7 & 3 \end{pmatrix}$

12. $\begin{pmatrix} 1 & 2 & 3 \\ 1 & 1 & 2 \\ 0 & 1 & 2 \end{pmatrix}$

13. $\begin{pmatrix} 1 & 1 & 1 & 1 \\ 1 & 2 & -1 & 2 \\ 1 & -1 & 2 & 1 \\ 1 & 3 & 3 & 2 \end{pmatrix}$

14. $\begin{pmatrix} 1 & 0 & 2 & 3 \\ -1 & 1 & 0 & 4 \\ 2 & 1 & -1 & 3 \\ -1 & 0 & 5 & 7 \end{pmatrix}$

15. $\begin{pmatrix} 1 & -3 & 0 & -2 \\ 3 & -12 & -2 & -6 \\ -2 & 10 & 2 & 5 \\ -1 & 6 & 1 & 3 \end{pmatrix}$

16. Show that if A, B, and C are invertible matrices, then ABC is invertible and $(ABC)^{-1} = C^{-1}B^{-1}A^{-1}$.

17. If $A_1, A_2, \ldots, A_m$ are invertible $n \times n$ matrices, show that $A_1A_2 \ldots A_m$ is invertible and calculate its inverse.

18. Show that the matrix $\begin{pmatrix} 3 & 4 \\ -2 & -3 \end{pmatrix}$ is equal to its own inverse.

19. Show that the matrix $A = \begin{pmatrix} a_{11} & a_{12} \\ a_{21} & a_{22} \end{pmatrix}$ is equal to its own inverse if $A = \pm I$ or if $a_{11} = -a_{22}$ and $a_{21}a_{12} = 1 - a_{11}^2$.

20. Find the output vector $\mathbf{x}$ in the Leontief input-output model if $n = 3$, $\mathbf{e} = \begin{pmatrix} 30 \\ 20 \\ 40 \end{pmatrix}$, and

$$A = \begin{pmatrix} \frac{1}{5} & \frac{1}{5} & 0 \\ \frac{2}{5} & \frac{2}{5} & \frac{3}{5} \\ \frac{1}{5} & \frac{1}{10} & \frac{2}{5} \end{pmatrix}.$$

***21.** Suppose that A is $n \times m$ and B is $m \times n$ so that AB is $n \times n$. Show that AB is not invertible if $n > m$. [*Hint:* Show that there is a nonzero vector $\mathbf{x}$ such that $AB\mathbf{x} = \mathbf{0}$ and then apply Theorem 6.]

***22.** Use the methods of this section to find the inverses of the following matrices with complex entries:

a. $\begin{pmatrix} i & 2 \\ 1 & -i \end{pmatrix}$

b. $\begin{pmatrix} 1-i & 0 \\ 0 & 1+i \end{pmatrix}$

c. $\begin{pmatrix} 1 & i & 0 \\ -i & 0 & 1 \\ 0 & 1+i & 1-i \end{pmatrix}$

23. Show that for every real number θ the matrix $\begin{pmatrix} \sin\theta & \cos\theta & 0 \\ \cos\theta & -\sin\theta & 0 \\ 0 & 0 & 1 \end{pmatrix}$ is invertible and find its inverse.

24. Calculate the inverse of $A = \begin{pmatrix} 2 & 0 & 0 \\ 0 & 3 & 0 \\ 0 & 0 & 4 \end{pmatrix}$.

25. A square matrix $A = (a_{ij})$ is called **diagonal** if all its elements off the main diagonal are zero. That is, $a_{ij} = 0$ if $i \neq j$. (The matrix of Problem 24 is diagonal.) Show that a diagonal matrix is invertible if and only if each of its diagonal components is nonzero.

26. Let

$$A = \begin{pmatrix} a_{11} & 0 & \cdots & 0 \\ 0 & a_{22} & \cdots & 0 \\ \vdots & \vdots & \ddots & \vdots \\ 0 & 0 & \cdots & a_{nn} \end{pmatrix}$$

be a diagonal matrix such that each of its diagonal components is nonzero. Calculate A^{-1}.

27. Calculate the inverse of $A = \begin{pmatrix} 2 & 1 & -1 \\ 0 & 3 & 4 \\ 0 & 0 & 5 \end{pmatrix}$.

28. Show that the matrix $A = \begin{pmatrix} 1 & 0 & 0 \\ -2 & 0 & 0 \\ 4 & 6 & 1 \end{pmatrix}$ is not invertible.

29. A square matrix is called **upper (lower) triangular** if all its elements below (above) the main diagonal are zero. (The matrix of Problem 27 is upper triangular and the matrix of Problem 28 is lower triangular.) Show that an upper or lower triangular matrix is invertible if and only if each of its diagonal elements is nonzero.

30. Show that the inverse of an invertible upper triangular matrix is upper triangular. [*Hint:* First prove the result for a 3×3 matrix.]

In Problems 31 and 32 a matrix is given. In each case show that the matrix is not invertible by finding a nonzero vector $\mathbf{x}$ such that $A\mathbf{x} = \mathbf{0}$.

31. $\begin{pmatrix} 2 & -1 \\ -4 & 2 \end{pmatrix}$

32. $\begin{pmatrix} 1 & -1 & 3 \\ 0 & 4 & -2 \\ 2 & -6 & 8 \end{pmatrix}$

33. A factory for the construction of quality furniture has two divisions: a machine shop where the parts of the furniture are fabricated, and an assembly and finishing division where the parts are put together into the finished product. Suppose that there are 12 employees in the machine shop and 20 in the assembly and finishing division and that each employee works an 8-hour day. Suppose further that the factory produces only two products: chairs and tables. A chair requires $\frac{384}{17}$ hours of machine shop time and $\frac{480}{17}$ hours of assembly and finishing time. A table requires $\frac{240}{17}$ hours of machine shop time and $\frac{640}{17}$ hours of assembly and finishing time. Assuming that there is an unlimited demand for these products and that the manufacturer wishes to keep all employees busy, how many chairs and how many tables can this factory produce each day?

34. A witch's magic cupboard contains 10 ounces of ground four-leaf clovers and 14 ounces of powdered mandrake root. The cupboard will replenish itself automatically provided she uses up exactly all her supplies. A batch of love potion requires $3\frac{1}{13}$ ounces of ground four-leaf clovers and $2\frac{3}{13}$ ounces of powdered mandrake root. One recipe of a well-known (to witches) cure for the common cold requires $5\frac{5}{13}$ ounces of four-leaf clovers and $10\frac{10}{13}$ ounces of mandrake root. How much of the love potion and the cold remedy should the witch make in order to use up the supply in the cupboard exactly?

35. A farmer feeds his cattle a mixture of two types of feed. One standard unit of type A feed supplies a steer with 10% of its minimum daily requirement of protein and 15% of its requirement of carbohydrates. Type B feed contains 12% of the requirement of protein and 8% of the requirement of carbohydrates in a standard unit. If the farmer wishes to feed his cattle exactly 100% of their minimum daily requirement of protein and carbohydrates, how many units of each type of feed should he give a steer each day?

36. A much simplified version of an input-output table for the 1958 Israeli economy divides that economy into three sectors—agriculture, manufacturing, and energy—with the following result.[†]

	Agriculture	Manufacturing	Energy
Agriculture	0.293	0	0
Manufacturing	0.014	0.207	0.017
Energy	0.044	0.010	0.216

a. How many units of agricultural production are required to produce one unit of agricultural output?

b. How many units of agricultural production are required to produce 200,000 units of agricultural output?

c. How many units of agricultural product go into the production of 50,000 units of energy?

d. How many units of energy go into the production of 50,000 units of agricultural products?

[†] Wassily Leontief, *Input-Output Economics* (New York: Oxford University Press, 1966), 54–57.